11. CLASS EQUATIONS

§11.1. Babylonian Equations

If we want to investigate all the character tables of a given size we need to find all groups with a given number of conjugacy classes. We shall see that there are only finitely many possible orders for a group with k conjugacy classes, and hence only finitely many $k \times k$ character tables for any k. For small values of k it is possible to catalogue them.

The **class equation** of a finite group is:
$$|G| = n_1 + n_2 + \ldots n_k$$
where the n_i's are the sizes of the conjugacy classes and
$$n_1 \leq n_2 \leq \ldots \leq n_k.$$
When there are c classes of a given size n, we often write $n*c$ instead of $n + n + \ldots + n$ (c times)

Example 1:
The class equation for $\mathbf{S}_4$ is $24 = 1 + 3 + 6*2 + 8$ and for $\mathbf{D}_{16}$ it is $16 = 1*2 + 2*3 + 4*2$.

Often the class equation completely characterises the group, but there are some groups that share the same class equation.

Example 2:
Both $\mathbf{D}_8 = \langle A, B \mid A^4 = B^2 = 1, BA = A^{-1}B \rangle$ and
$\mathbf{Q}_8 = \langle A, B \mid A^4 = 1, B^2 = A^2, BA = A^{-1}B \rangle$ have the same
class equation: $8 = 1*2 + 2*3$.

Now each n_i is the index of the corresponding centraliser
in G and so divides |G|. If we divide a class equation by
|G| we get an equation of the form:
$$1 = \frac{1}{m_1} + \dots + \frac{1}{m_k}$$
where each m_i is a positive integer.

A **Babylonian equation** is an equation of the form:
$$1 = \frac{1}{m_1} + \dots + \frac{1}{m_k}$$
where each $m_i \in \mathbb{Z}^+$ and $m_1 \le m_2 \le .. \le m_k$. The **length** of
such an equation is k.

Example 3:
The class equation for $\mathbf{S}_4$ is $24 = 1 + 3 + 6 + 6 + 8$ which
gives the Babylonian equation: $1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{24}$.

Theorem 1: There are only finitely many Babylonian
equations of a given length.
Proof: For a Babylonian equation of length k:
$$1 = \frac{1}{m_1} + \dots + \frac{1}{m_k}$$

suppose we have proved that there are finitely many choices for $m_1, m_2, \ldots m_i$.

Let $i < k$ and let $M = 1 - \dfrac{1}{m_1} - \dfrac{1}{m_2} - \ldots - \dfrac{1}{m_i}$.

Then $\dfrac{1}{m_{i+1}} + \ldots + \dfrac{1}{m_k} = M$ and we have finitely many choices for M.

Since $\dfrac{1}{m_{i+1}} \geq \ldots \geq \dfrac{1}{m_k}$ we have $M \leq \dfrac{k-i}{m_{i+1}}$.

So $\dfrac{1}{1-M} < m_{i+1} \leq \dfrac{k-i}{M}$, giving only finitely many choices for m_{i+1}.

Corollary: There are only finitely many class equations of a given length and hence only finitely many character tables of a given size.

Example 3: Babylonian equations of length ≤ 4:

Length 1
$1 = 1$

Length 2
$1 = \dfrac{1}{2} + \dfrac{1}{2}$

Length 3		
$1 = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{6}$	$1 = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{4}$	$1 = \dfrac{1}{3} + \dfrac{1}{3} + \dfrac{1}{3}$

Length 4	
$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42}$	$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24}$
$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15}$	$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18}$
$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{12}$	$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$
$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$	$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}$
$1 = \frac{1}{2} + \frac{1}{5} + \frac{1}{5} + \frac{1}{10}$	$1 = \frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$	$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6}$
$1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6}$	$1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$

Example 4: Class equations of length ≤ 3

Babylonian	Class Eqn	Groups
$1 = 1$	$1 = 1$	1
$1 = \frac{1}{2} + \frac{1}{2}$	$2 = 1 + 1$	$\mathbf{C}_2$
$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$	$6 = 1 + 2 + 3$	$\mathbf{S}_3 = \mathbf{D}_6$
$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$	$4 = 1 + 1 + 2$	none
$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$	$3 = 1 + 1 + 1$	$\mathbf{C}_3$

§11.2. Some Elementary Tests for Potential Class Equations

Much of the material in this chapter arose in the 1980s from my teaching a course on group theory.

It's a routine exercise to generate all Babylonian equations of a given length and to obtain a list of possible class equations. The problem is to exclude those that do not arise. A subsequent problem is to identify the groups that give rise to the potential class equations.

We'll focus on the first problem by investigating properties that class equations must satisfy. Each such property will give rise to a test. Then when we have a potential class equation we put it through the battery of tests. Only if it survives do we go in search for possible groups. If, after a reasonable amount of effort, we can find no such group we might then look for another reason to exclude that class equation. The set of tests we shall develop will catch the vast majority of false class equations – quite probably no such set of tests will ever catch them all, for as the number of conjugacy classes increases there appear to be more and more subtle reasons that exclude a given class equation.

Theorem 2 (Z Test): Suppose $|G| = N$ and $|Z(G)| = m$. If the class equation for G is

$$N = 1*m + n_{m+1} + n_{m+2} + \ldots + n_m$$

where each $n_{m+1} > 1$ then m must properly divide N/n_i for each i.

Proof: Suppose $g \notin Z(G)$.

Then $|C_G(g)| = N/n_i$ for some $i > m$.

Since $g \in C_G(g)$, $Z(G)$ must be a proper
subgroup of $C_G(g)$ and so m properly divides N/n_i.

Example 5: $84 = 1 + 1 + 12 + 21 + 21 + 28$ is not a class
equation.

Theorem 3 (*pq* Test): Suppose
$$N = 1 + n_2 + n_3 + \ldots + n_k$$
is the class equations for a group G and, for some $i > 1$,
$N/n_i = p^a q^b$, where p, q are distinct
 primes and $a, b \geq 1$.

Then the number of j for which pq divides N/n_j is at least
4.

Proof: We need to find four conjugacy classes where the
orders of the centralisers are all divisible by pq.

Suppose $|C_G(g)| = p^a q^b$ where $g \neq 1$ and suppose that g
has order d.

Then either p or q divides d.

Suppose, without loss of generality, p divides d, in which
case some power of g has order p.

By Cauchy's theorem $C_G(g)$ contains an element of order
q, which commutes with that power of order p and so G
has an element of order pq and so elements of order 1, p,
q and pq whose centralisers have orders divisible by pq.

As the orders are different they must belong to distinct
conjugacy classes.

Example 6: $120 = 1 + 5 + 20 + 24 + 30 + 40$ is not a class equation because if it was the respective centralisers would have orders 120, 24, 6, 5, 4 and 3.

Theorem 4 (pN Test): Suppose p is prime and
$$pN = 1 + n_2 + n_3 + \ldots + n_{k-t} + N*t$$
is the class equation for a group G where $N > 1$, $t \geq 1$ and $n_{k-t} < N$.
Then: (i) $t \mid p - 1$ and

(ii) $N \equiv \dfrac{p-1}{t} \pmod{p}$.

Proof: Let Γ be a class of size N and let $g \in \Gamma$. Then $|C_G(g)| = p$ and so $C_G(g) = \langle g \rangle$.
If $g^s \neq 1$ then $C_G(g^s) = \langle g \rangle$ and so g^s lies in a conjugacy class of size N.
Let Ω be any conjugacy class of size N and let $\langle g \rangle$ act on it by conjugation.
The orbits have sizes 1 or p. But orbits of size 1 correspond to non-trivial powers of g, and there are $p - 1$ of these altogether, so the number of orbits of size 1 in Γ is at most $p - 1$.
Now if $N = np + r$ where $0 \leq r < p$ there must be exactly r orbits of size 1 in Ω.
So there are exactly r powers of g in each of the t conjugacy classes of size N and hence
$p - 1 = rt$. Hence $N \equiv r = \dfrac{p-1}{t} \pmod{p}$.

Example 7: $216 = 1 + 8 + 27 + 54 + 54 + 72$ is not a class equation.

Here $p = 3$, $t = 1$ and $N = 72 \equiv 0 \pmod 3$.

§11.3. The 2N Test

The largest possible size of a conjugacy class, for a non-trivial group of order M, is M/2. Of course there can only be one of these and its elements must have order 2. In fact the class equation of such a group is completely determined by this property.

Theorem 5 (2N test): Let $|G| = 2N$ and let Γ be a conjugacy class of size N.

Then N is odd and the class equation for G is

$$2N = 1 + 2 + 2 + \ldots + 2 + N = 1 + 2*\frac{N-1}{2}.$$

Proof:

(1) The elements of Γ have order 2 and commute only with 1 and themselves:

This is because the centralisers of these elements have order 2N/N.

(2) H = G − Γ is a normal subgroup of G:

H clearly contains 1 and is closed under inverses. Let x, y be distinct elements of Γ.

If $xy \in \Gamma$ then $(xy)^2 = x^2y^2 = 1$, so $xy = yx$ and hence $y \in C_G(x)$, a contradiction. Hence H is a subgroup of G and, being a subgroup of index 2, it is a normal subgroup.

	H	**Γ**
H	H	K
Γ	Γ	H

(3) H is abelian:
Let $h \in$ H and $k \in \Gamma$. If $hk \in$ H then $k \in$ H.
Hence $hk \in \Gamma$ and so $(hk)^2 = 1$.
Thus $k^{-1}hk = h^{-1}$ for all $h \in$ H and so $h \rightarrow h^{-1}$ is an automorphism of H.
If $h_1, h_2 \in H$ then $(h_1 h_2)^{-1} = h_1^{-1} h_2^{-1} = (h_2 h_1)^{-1}$ and so $h_1 h_2 = h_2 h_1$.

(4) N = |H| is odd.
If |H| is even then H would contain an element h of order 2. Since $h = h^{-1}$ it follows that h commutes with k, a contradiction.

(5) The class equation for G is
$$2N = 1 + 2 + 2 + \ldots + 2 + N.$$
Since $k^{-1}hk = h^{-1}$ for all $k \in$ K and all non-trivial $h \in$ H, the conjugacy classes, apart from $\{1\}$ and Γ, are all of the form $\{h, h^{-1}\}$.

(6) N is odd.

Example 8: $18 = 1 + 2 + 6 + 9$ is not a class equation.

§11.4. The 3N Test
Lemma: If $z = a + b\omega$ where $a, b \in \mathbb{Z}$ then
$$|z|^2 \in \mathbb{Z}.$$
Proof: Multiplying z by its conjugate we get
$|z|^2 = (a + b\omega)(a + b\omega^2)$
$\quad = a^2 + b^2 + ab\omega + ab\omega^2$
$\quad = a^2 + b^2 - ab \in \mathbb{Z}.$

Theorem 6 (3N Test): Suppose $|G| = 3N$ and G has precisely 2 classes of size N.
Then $|G'| = N$ and the class equation for G is
$$3N = 1 + 3t_1 + 3t_2 + \ldots + 3t_k + N*2 \text{ where}$$
$$N = 1 + t_1*3 + t_2*3 + \ldots + t_k*3$$
is the class equation for G'.
Proof: The elements of Γ have order 3 and commute only with their powers.
Suppose Γ is a class of size N.
If $\Gamma = \Gamma^{-1}$ then there exists $g \in \Gamma$ and $x \in G$ such that
$$x^{-1}gx = g^{-1}.$$
Then $x^2 \in C_G(g)$.
But $|C_G(g)| = 3$ so $x \in C_G(g)$.
Hence $g^2 = 1$, a contradiction.
Hence the two conjugacy classes of size N are Γ and Γ^{-1}.
By column orthogonality there must be a non-real entry in the Γ column of the character table for G and, since the eigenvalues of the corresponding matrix must be cube

roots of 1, this entry must have the form $a + b\omega$ where a, b are integers and $b \neq 0$.

The character table for G contains the sub-table:

class	**1**	Γ	Γ^{-1}
size	**1**	**N**	**N**
χ_1	1	1	1
χ_2	n	$a + b\omega$	$a + b\omega^2$
χ_3	n	$a + b\omega^2$	$a + b\omega$
order	**1**	**3**	**3**

where $a, b \in \mathbb{Z}$ with $b \neq 0$.

By the lemma, $|a + b\omega|^2 = |a + b\omega^2|^2$ are positive integers and since the sum of squares of the entries in each of the last two columns is $3N/N = 3$, we must have
$$|a + b\omega|^2 = |a + b\omega^2|^2 = 1$$
and all other entries in these columns must be zero.

So, by orthogonality with the first column,
$0 = 1 + n(a + b\omega) + n(a + b\omega^2)$
$\quad = 1 + n(2a - b)$.
Hence $n = 1$.
Thus G has at least 3 linear characters.

But if χ is any linear character then $\chi(\Gamma) \neq 0$ and so G has exactly 3 linear characters and so $|G'| = N$.
Clearly $G' = G - \Gamma - \Gamma^{-1}$.

The elements of $\Gamma + \Gamma^{-1}$ have centralisers of order 3 so can't commute with any non-trivial element of G′.

Hence if $1 \neq g \in G'$, $C_{G'}(g) = C_G(g)$.
Thus if $g \in G'$ has $t > 1$ conjugates in G′ then it has $3t$ conjugates in G.

Example 8: If G has class equation
$$48 = 1 + 3 + 12 + 16 + 16$$
then G′ has class equation
$$16 = 1 + 1 + 1 + 1 + 4 + 4 + 4,$$
which fails the Z Test.

EXERCISES FOR CHAPTER 11

Exercise 1: For each of the following statements determine whether it is true or false.
(1) There is a group with class equation:
$$72 = 1 + 2 + 3 + 3 + 6 + 8 + 9 + 16 + 24.$$
(2) There is a group with class equation:
$$32 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 8 + 8 + 8.$$
(3) There is a group with class equation:
$$32 = 1 + 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 16.$$
(4) There is a group with class equation:
$$120 = 1 + 1 + 15 + 15 + 20 + 20 + 24 + 24.$$
(5) There is a group with class equation:
$$70 = 1 + 2 + 2 + 7 + 7 + 7 + 7 + 35$$
(6) There is a group with class equation:
$$60 = 1 + 2 + 5 + 12 + 20 + 20.$$
(7) There is a group with class equation:
$$50 = 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 25$$
(8) There is a group with class equation:
$$48 = 1 + 3 + 4 + 8 + 16 + 16.$$

Exercise 2: For the Babylonian equations of length 4 given above, write down the corresponding class equations. Eliminate as many as possible and identify possible groups for as many of the remaining ones that you can.

Exercise 3: Find all the Babylonian equations of length 5 where each term is less than ½.

SOLUTIONS FOR CHAPTER 11

Exercise 1:
(1) FALSE: 16 does not divide 72.
(2) FALSE: For such a group G, $|Z(G)| = 8$.
The centralisers of the other groups have order 4, but they must contain the centre, a contradiction.
(3) TRUE: It is $\mathbf{D}_{32}$.
(4) TRUE: For such a group G, $|Z(G)| = 2$.
Let $Z(G) = \langle A \rangle$. Suppose that $G = H \times \mathbf{C}_2$. If so, the class equation for H would be $60 = 1 + 15 + 20 + 24$, which we recognise as the class equation of A_5. So $G = \mathbf{A}_5 \times \mathbf{C}_2$ has this class equation.
(5) FALSE: By the 2N test.
(6) FALSE: By the 3N test.
(7) TRUE: Such a group is D_{50}
(8) FALSE: By the 3N test.

Exercise 2:

CLASS EQUATION	COMMENTS
$42 = 1 + 6 + 14 + 21$	2N test
$24 = 1 + 3 + 8 + 12$	2N test
$15 = 1 + 1.5 + 5 + 7.5$	Sizes of CC must be integers
$18 = 1 + 2 + 6 + 9$	2N test
$12 = 1 + 1 + 4 + 6$	2N test
$20 = 1 + 4 + 5 + 10$	2N test
$12 = 1 + 2 + 3 + 6$	2N test
$8 = 1 + 1 + 2 + 4$	2N test

$10 = 1 + 2 + 2 + 5$	$\mathbf{D}_{10}$
$6 = 1 + 1 + 1 + 3$	2N test
$12 = 1 + 3 + 4 + 4$	3N test
$6 = 1 + 1 + 2 + 2$	3N test
$6 = 1 + 1.5 + 1.5 + 2$	Sizes of CC must be integers
$4 = 1 + 1 + 1 + 1$	$\mathbf{C}_4$, $\mathbf{V}_4$

Exercise 3:

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{13} + \frac{1}{156}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{14} + \frac{1}{84}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{15} + \frac{1}{60}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{16} + \frac{1}{48}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{18} + \frac{1}{36}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{20} + \frac{1}{30}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{21} + \frac{1}{28}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{24} + \frac{1}{24}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{8} + \frac{1}{120}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{45}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{10} + \frac{1}{30}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{12} + \frac{1}{20}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{5} + \frac{1}{15} + \frac{1}{15}$$

$$1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{7} + \frac{1}{42}$$

$$1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{24}$$

$$1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{18}$$

$$1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{10} + \frac{1}{15}$$

$$1 = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{12} + \frac{1}{12}$$

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{15}$$

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10}$$

$$1 = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20}$$

$$1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$$

$$1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}$$

$$1 = \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \frac{1}{5} + \frac{1}{10}$$

$$1 = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$$

You might like to work out the corresponding class equations and eliminate those that are not possible and identify any you recognise.